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An approximate partition function of the Ising model on the Sierpinski gasket

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Abstract. The partition function of the Ising model on the Sierpinski gasket is obtained by using a decoupled Sierpinski gasket Ising model as an unperturbed system. The exact partition function can be expressed as a statistical average of a special product over an unperturbed Ising Hamiltonian. A high-temperature expansion of the partition function is produced.

1. Introduction

The Sierpinski gasket as a regular fractal (an exact mathematical fractal) has been extensively studied (Gefen *et al* 1980, 1981, 1984). Much of the current interest in such systems focuses on the influence of the geometrical structure on the behaviour of phase transition. Recently, Gefen *et al* have proved with a decimation RG technique that no finite-temperature transition exists in such systems due to their finite order of ramification. Also for the same reason, we expect that all spin models on the Sierpinski gasket can be exactly solved, in principle. However, no exact result related partition function has been done so far because of the existence of a great deal of spin correlations, which makes calculation complicated.

In this paper we propose a calculation of the partition function for the Ising model on a Sierpinski gasket embedded in two-dimensional Euclidean space. We define a decoupled Sierpinski gasket (DSG) Ising model as an unperturbed system and the exact partition function of the Sierpinski gasket (SG) Ising model can be expressed as a statistical average of a special product over an unperturbed DSG Ising Hamiltonian. As an approximation, a high-temperature expansion of partition function is also given.

The paper is organised as follows. In § 2, we give the Ising Hamiltonian of the second construction stage ($N=2$) of the SG and define a decoupled Sierpinski gasket corresponding to the $N=2$ stage of the construction of the SG. In § 3 we give the expression of the partition function of the Nth construction stage of SG. Section 4 calculates the partition function of DSG. Section 5 calculates the statistical average of various types over the DSG Ising Hamiltonian. Finally, § 6 gives a conclusion and discussion.

2. The SG Ising Hamiltonian and DSG Ising Hamiltonian

Consider the structure shown in figure 1(a), which is the second construction stage of the SG. Let us assign an Ising spin on each site. We write the reduced Ising Hamiltonian

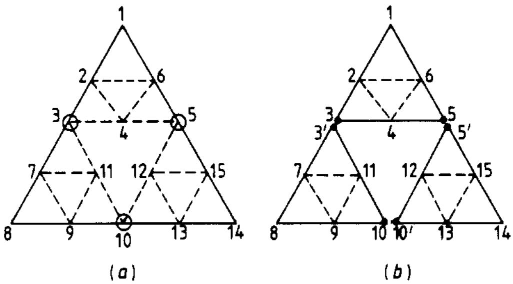


Figure 1. The SG and DSG in 2D. The second construction stage is shown. In (a) sites 3, 5 and 10 are decoupled and then become (b). Figure (a) is the SG and figure (b) is the DSG.

as follows

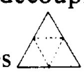
$$-\beta\mathscr{H}_{\mathrm{SG}} = H_{\mathrm{SG}} = \sum_{\langle i=j\rangle} K s_i s_j \tag{1}$$

where summation is over all nearest-neighbour spins and K is a coupling parameter associated with exchange integral J with $K = \beta J = J/kT$.

We now cut this structure at sites 3, 5 and 10, which means that a decoupling procedure has taken place, which then results in a DSG Ising model as in figure 1(b). The Ising Hamiltonian of DSG is given by

$$-\beta\mathscr{H}_{\mathrm{DSG}} = H_{\mathrm{DSG}} = \sum K s_{i_1} s_{j_1} + \sum K s_{i_2} s_{j_2} + \sum K s_{i_3} s_{j_3} \tag{2}$$

where summations are over all nearest-neighbour spins on isolated triangles, and s_{i_1}, s_{i_2} and s_{i_3} denote lattice site spins on three isolated triangles of DSG. Hereafter we regard DSG as an unperturbed system and SG as a perturbed system. For our purposes we will not write out the perturbation Hamiltonian H' explicitly.

In general, the Nth stage structure of DSG can be produced in terms of the decoupling procedure, the decoupled sites are between two nearest-neighbour triangles . It is easy to find that the number of decoupled sites can be expressed as

$$n = \tfrac{1}{2}(3^N - 3) \tag{3}$$

where N denotes the order of the stage of structure for SG and DSG.

3. Partition function

In order to be explicit we first give the partition function of the $N = 2$ stage of structure for SG (see figure 1(a))

$$\begin{aligned} Z_{\mathrm{SG}}(N=2) &= \sum_{\{s\}} \mathrm{e}^{H_{\mathrm{SG}}} = \sum_{\{s\}} \exp[K(s_1 s_2 + s_2 s_3 + \ldots + s_{14} s_{15})] \\ &= \sum_{[s]} (A_1 A_2 A_3)\, \mathrm{e}^{H_r} \end{aligned} \tag{4}$$

where $\{s\}$ represents a set of site spins $(s_1, s_2, \ldots, s_{14}, s_{15})$ and $[s]$ also represents a set of site spins, but s_3, s_5 and s_{10} are excluded. A_1, A_2 and A_3 are products summing over s_3, s_5 and s_{10}, their expressions are respectively

$$\begin{aligned} A_1 &= \exp[K(s_2 + s_4 + s_7 + s_{11})] + \exp[-K(s_2 + s_4 + s_7 + s_{11})] \\ A_2 &= \exp[K(s_4 + s_6 + s_{12} + s_{15})] + \exp[-K(s_4 + s_6 + s_{12} + s_{15})] \end{aligned} \tag{5}$$

and

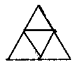
$$A_3 = \exp[K(s_9 + s_{11} + s_{12} + s_{13})] + \exp[-K(s_9 + s_{11} + s_{12} + s_{13})].$$

H_r is the remainder of the SG Ising Hamiltonian after summing over s_3, s_5 and s_{10}, which can be written (in the $N = 2$ case) as

$$H_r(N=2) = K(s_1s_2 + s_1s_6 + s_2s_4 + s_4s_6 + s_2s_6) + K(s_7s_8 + s_8s_9 + s_7s_9 + s_9s_{11} + s_7s_{11}) + K(s_{13}s_{14} + s_{14}s_{15} + s_{12}s_{13} + s_{13}s_{15} + s_{12}s_{15}). \tag{6}$$

Similarly, the partition function of DSG is

$$Z_{\mathrm{DSG}}(N=2) = \sum_{\{s_\mathrm{D}\}} \mathrm{e}^{H_{\mathrm{DSG}}} = \sum_{[s]} (B_1B_2B_3)\, \mathrm{e}^{H_r} \tag{7}$$

where $\{s_\mathrm{D}\}$ includes all site spins of three isolated triangles , and $[s]$ includes all except s_3, $s_{3'}$, s_5, $s_{5'}$ and s_{10}, $s_{10'}$ site spins. B_1, B_2 and B_3 are respectively

$$B_1 = \exp[K(s_2 + s_4 + s_7 + s_{11})] + \exp[-K(s_2 + s_4 + s_7 + s_{11})] + \exp[K(s_2 + s_4 - s_7 - s_{11})] + \exp[-K(s_2 + s_4 - s_7 - s_{11})]$$

$$B_2 = \exp[K(s_4 + s_6 + s_{12} + s_{15})] + \exp[-K(s_4 + s_6 + s_{12} + s_{15})] + \exp[K(s_4 + s_6 - s_{12} - s_{15})] + \exp[-K(s_4 + s_6 - s_{12} - s_{15})] \tag{8}$$

and

$$B_3 = \exp[K(s_9 + s_{11} + s_{12} + s_{13})] + \exp[-K(s_9 + s_{11} + s_{12} + s_{13})] + \exp[K(s_9 + s_{11} - s_{12} - s_{13})] + \exp[-K(s_9 + s_{11} - s_{12} - s_{13})].$$

The expression for H_r is the same as (6).

We will rewrite Z_{SG} as

$$Z_{\mathrm{SG}}(N=2) = \sum_{[s]} (A_1A_2A_3)\, \mathrm{e}^{H_r} = \sum_{[s]} (B_1B_2B_3)\, \mathrm{e}^{H_r} \left(\frac{A_1}{B_1}\frac{A_2}{B_2}\frac{A_3}{B_3}\right). \tag{9}$$

It will be found that

$$\frac{Z_{\mathrm{SG}}(N=2)}{Z_{\mathrm{DSG}}(N=2)} = \sum_{\{s_\mathrm{D}\}} \mathrm{e}^{H_{\mathrm{DSG}}} \left(\frac{A_1}{B_1}\frac{A_2}{B_2}\frac{A_3}{B_3}\right)\left(\sum_{\{s_\mathrm{D}\}} \mathrm{e}^{H_{\mathrm{DSG}}}\right)^{-1} = \left\langle\frac{A_1}{B_1}\frac{A_2}{B_2}\frac{A_3}{B_3}\right\rangle_{\mathrm{DSG}} \tag{10}$$

where $\langle(A_1/B_1), (A_2/B_2), (A_3/B_3)\rangle_{\mathrm{DSG}}$ is the statistical average of the product (A_1/B_1), (A_2/B_2), (A_3/B_3) over the unperturbed DSG Ising Hamiltonian. In general, for stage N of the structure for SG and DSG we have

$$\frac{Z_{\mathrm{SG}}(N)}{Z_{\mathrm{DSG}}(N)} = \left\langle\frac{A_1}{B_1}\frac{A_2}{B_2}\cdots\frac{A_n}{B_n}\right\rangle_{\mathrm{DSG}} \tag{11}$$

where n is the number of decoupled sites, which is given by (3). The expression (11) is our essential starting point. However, for convenience of calculation we introduce a quantity α_i, which is defined as

$$\alpha_i = 1 - (A_i/B_i)$$

and can be expressed in the $N = 2$ case, for example, as

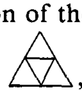
$$\alpha_1 = \frac{\cosh K(s_2 + s_4 - s_7 - s_{11})}{\cosh K(s_2 + s_4 + s_7 + s_{11}) + \cosh K(s_2 + s_4 - s_7 - s_{11})}. \tag{12}$$

It is noted that all site spins in (12) are nearest neighbours of decoupled site spin s_3 (see figure 1(a)). In terms of α_i we obtain the following expression

$$\left\langle \frac{A_1}{B_1}\frac{A_2}{B_2}\cdots\frac{A_n}{B_n}\right\rangle_{\mathrm{DSG}} = \langle(1-\alpha_1)(1-\alpha_2)\ldots(1-\alpha_n)\rangle_{\mathrm{DSG}}$$
$$= 1-\langle\alpha_1\rangle-\langle\alpha_2\rangle-\ldots\langle\alpha_n\rangle+\langle\alpha_1\alpha_2\rangle+\langle\alpha_2\alpha_3\rangle+\ldots$$
$$+\langle\alpha_{n-1}\alpha_n\rangle+\ldots+\langle\alpha_1\alpha_2\ldots\alpha_n\rangle. \tag{13}$$

In this expression the number of terms of $\langle\alpha_i\rangle$-type, $\langle\alpha_i\alpha_j\rangle$-type, ... and $\langle\alpha_1\alpha_2\ldots\alpha_n\rangle$-type are respectively n, $n(n-1)/2!$, $n(n-1)(n-2)/3!$, ... and 1.

4. Partition function of the DSG Ising model

Before we proceed to calculate the statistical average mentioned above, we first find the partition function of the DSG Ising model. Since a DSG is composed of a number of isolated triangles , it is sufficient to find the partition function of a triangle shown in figure 2. By using a decimation procedure and transfer matrix method, we obtain the partition function as follows

$$Z_0 = (2\cosh K')^3 + (2\sinh K')^3 \tag{14}$$

where K' is the renormalised coupling parameter which is the consequence of carrying out a decimation operation for s_1, s_3 and s_5. A scheme of such a procedure is shown in figure 3. It is easy to find that

$$K' = K + \tfrac{1}{2}\ln(\cosh 2K). \tag{15}$$

Finally, we obtain the partition function of the DSG Ising model

$$Z_{\mathrm{DSG}} = Z_0^{3^{N-1}} \tag{16}$$

where N is the order of the construction stage.

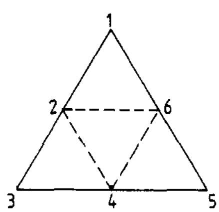


Figure 2. A basic triangle of DSG is shown.

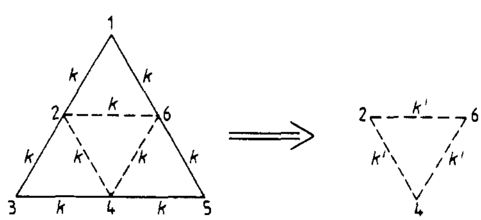


Figure 3. A scheme of decimation procedure is shown. K is the original coupling parameter and K' is the renormalised coupling parameter.

5. Calculation of statistical average values

We now proceed to calculate statistical averages $\langle\alpha_i\rangle$, $\langle\alpha_i\alpha_j\rangle$ and $\langle\alpha_i\alpha_j\alpha_k\rangle$.

5.1. *To find* $\langle\alpha_i\rangle$

By definition, we have, for example,

$$\langle\alpha_1\rangle = \frac{1}{Z_{\mathrm{DSG}}}\sum_{\{s_{\mathrm{D}}\}} \mathrm{e}^{H_{\mathrm{DSG}}} \frac{\cosh K(s_2+s_4-s_7-s_{11})}{\cosh K(s_2+s_4+s_7+s_{11})+\cosh K(s_2+s_4-s_7-s_{11})} \tag{17}$$

where s_2, s_4 and s_7, s_{11} come from two nearest-neighbour triangles respectively, they are associated with site spin s_3. Employing the series expansion of $\cosh x$, we can write, when $K<1$,

$$\frac{\cosh K(s_2+s_4-s_7-s_{11})}{\cosh K(s_2+s_4+s_7+s_{11})+\cosh K(s_2+s_4+s_7+s_{11})}$$
$$=\frac{1}{2}-\left(\frac{K^2}{2}+2K^4\right)(s_2s_7+s_2s_{11}+s_4s_7+s_4s_{11})+\mathrm{O}(K^6) \tag{18}$$

where $s_i^2=1$ is used. Substitute (18) into (17) and note that s_2, s_4 and s_7, s_{11} belong to different isolated triangles respectively. Thus we obtain the following result

$$\langle\alpha_1\rangle=\tfrac{1}{2}-(\tfrac{1}{2}K^2+2K^4)(\langle s_2\rangle\langle s_7\rangle+\langle s_2\rangle\langle s_{11}\rangle+\langle s_4\rangle\langle s_7\rangle+\langle s_4\rangle\langle s_{11}\rangle)+\mathrm{O}(K^6)$$
$$=\tfrac{1}{2} \tag{19}$$

where $\langle s_i\rangle=0$ has been used. We should point out that at first sight the result in (19) seems to be approximate, however, we will prove that it is really exact. In fact, a direct calculation shows us that

$$\langle\alpha_1\rangle=\frac{1}{Z_0^L}\sum_{\{s_{\mathrm{D}}\}}\mathrm{e}^{H_{\mathrm{DSG}}}\frac{\cosh K(s_2+s_4-s_7-s_{11})}{\cosh K(s_2+s_4+s_7+s_{11})+\cosh K(s_2+s_4-s_7-s_{11})}$$
$$=\frac{2\,\mathrm{e}^{6K'}+12\,\mathrm{e}^{2K'}+18\,\mathrm{e}^{-2K'}}{4\,\mathrm{e}^{6K'}+24\,\mathrm{e}^{2K'}+36\,\mathrm{e}^{-2K'}}=\frac{1}{2} \tag{20}$$

where $L=3^{N-1}$. In general, we have, for any i

$$\langle\alpha_i\rangle=\tfrac{1}{2}. \tag{21}$$

It is worth pointing out that in (20), due to the renormalised coupling parameter, K' is employed, so only 64 configurations of site spins have to be considered.

5.2. *To find* $\langle\alpha_i\alpha_j\rangle$

Here we have to distinguish two different cases: from case 1 α_i and α_j are associated with three isolated triangles, which means that one triangle is shared by two decoupled sites; in case 2, there are four isolated triangles, which means that no triangle is shared by two decoupled sites, i.e. they are far apart (see figure 4(*d*) α_i, α_k). First of all, let us consider case 1. By definition, we write

$$\langle\alpha_1\alpha_2\rangle=\frac{1}{Z_0^L}\sum_{\{s_{\mathrm{D}}\}}\mathrm{e}^{H_{\mathrm{DSG}}}\frac{\cosh K(s_2+s_4-s_7-s_{11})}{\cosh K(s_2+s_4+s_7+s_{11})+\cosh K(s_2+s_4-s_7-s_{11})}$$
$$\times\frac{\cosh K(s_4+s_6-s_{12}-s_{15})}{\cosh K(s_4+s_6+s_{12}+s_{15})+\cosh K(s_4+s_6-s_{12}-s_{15})}. \tag{22}$$

By using a series expansion of $\cosh x$ and employing $s_i^2=1$ and $\langle s_i\rangle=0$, we obtain

$$\langle\alpha_1\alpha_2\rangle=(\tfrac{1}{2})^2=\tfrac{1}{4}. \tag{23}$$

In general, we have

$$\langle\alpha_i\alpha_j\rangle=\tfrac{1}{4}. \tag{24}$$

We can also prove that this result is exact. In fact, a direct calculation shows us that

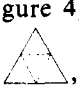
$$\langle\alpha_1\alpha_2\rangle=\frac{2\,\mathrm{e}^{9K'}+18\,\mathrm{e}^{5K'}+54\,\mathrm{e}^{K'}+54\,\mathrm{e}^{-3K'}}{8\,\mathrm{e}^{9K'}+72\,\mathrm{e}^{5K'}+216\,\mathrm{e}^{K'}+216\,\mathrm{e}^{-3K'}}=\frac{1}{4} \tag{25}$$

where 9 site spins and 2^9 configurations have to be treated after the decimation procedure for the corner spins of triangles has been carried out and then K is changed into K'. From case 2, since no correlation exists between α_i and α_j, they are independent statistically. As a consequence, we have

$$\langle\alpha_i\alpha_j\rangle=\langle\alpha_i\rangle\langle\alpha_j\rangle=\tfrac{1}{4}. \tag{26}$$

In summary, in both cases mentioned above $\langle\alpha_i\alpha_j\rangle$ equal the same value, $\frac{1}{4}$.

5.3. *To find* $\langle\alpha_i\alpha_j\alpha_k\rangle$

The calculation of $\langle\alpha_i\alpha_j\alpha_k\rangle$ is more complicated. However, we may classify configurations of related triangles by α_i, α_j and α_k according to their relative positions. The following five types of configuration will be possible (see figure 4). In figure 4(a), three decoupled site spins are associated with three triangles , which neighbour each other. Figures 4(b) and 4(c) display two types of configuration of triangles, in which three decoupled sites are associated with four triangles . Figure 4(d) consists of two independent parts, one of which includes three and the other includes two triangles. Finally, figure 4(e) is composed of three independent parts, in which each one includes two triangles and one decoupled site. First we consider figures 4(d) and 4(e). It is easy to find that

$$\langle\alpha_i\alpha_j\alpha_k\rangle=\langle\alpha_i\alpha_j\rangle\langle\alpha_k\rangle=(\tfrac{1}{2})^3=\tfrac{1}{8}$$

for figure 4(d), and

$$\langle\alpha_i\alpha_j\alpha_k\rangle=\langle\alpha_i\rangle\langle\alpha_j\rangle\langle\alpha_k\rangle=(\tfrac{1}{2})^3=\tfrac{1}{8}$$

for figure 4(e). Here (21) and (26) are used. Similarly, one finds that the contribution of figures 4(b) and 4(c) are the same as above.

Finally we calculate case (a) in figure 4, which will give a very different contribution to $\langle\alpha_i\alpha_j\alpha_k\rangle$. Calculation shows that in the expression of $\langle\alpha_i\alpha_j\alpha_k\rangle$ the following terms of four types are included.

(i) $\langle s_a^2s_b^2s_c^2\rangle=\langle s_a^2\rangle\langle s_b^2\rangle\langle s_c^2\rangle=1.$

(ii) $\langle s_a^2s_b^2s_cs_d\rangle=\langle s_cs_d\rangle=(\mathrm{e}^{3K'}-\mathrm{e}^{-K'})/(\mathrm{e}^{3K'}+3\mathrm{e}^{-K'})=c(K')$

if s_c and s_d belong to the same triangle.

(iii) $\langle s_a^2s_bs_cs_ds_e\rangle=\langle s_bs_cs_ds_e\rangle=\langle s_bs_c\rangle\langle s_ds_e\rangle=c^2(K')$

if s_b, s_c and s_d, s_e respectively belong to two different triangles.

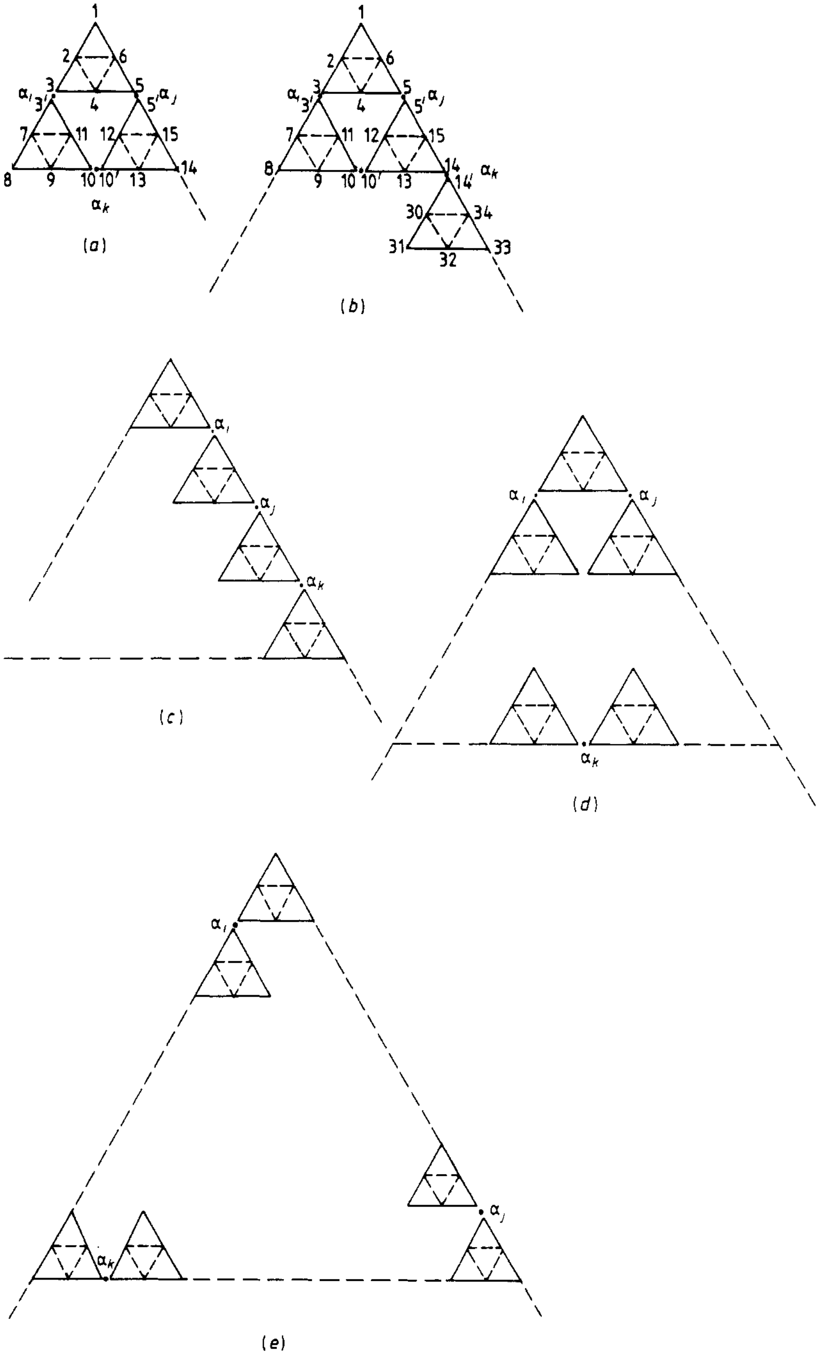


Figure 4. Five possible configurations of basic triangles are shown. Their description is given in the text.

(iv) $\langle s_a s_b s_c s_d s_e s_f\rangle = \langle s_a s_b\rangle\langle s_c s_d\rangle\langle s_e s_f\rangle = c^3(K')$

if s_a, s_b; s_c, s_d and s_e, s_f respectively belong to three different triangles.

Then we find that

$$\langle \alpha_i\alpha_j\alpha_k\rangle = (\tfrac{1}{2})^3 - (\tfrac{1}{2}K^2+2K^4)^3(1+9c(K')+27c^2(K')+27c^3(K'))+\ldots \tag{27}$$

Furthermore we have to compute the number of configurations like figure 4(a) included in the Nth construction stage of DSG. Obviously, it equals 3^{N-2}, and then there are $\{[n(n-1)(n-2)/3!]-3^{N-2}\}$ terms, in which each only gives a contribution of $(\frac{1}{2})^3$. It is worth noting that for the configuration shown in figure 4(a) a rigorous exact calculation gives the following result:

$$\langle \alpha_i\alpha_j\alpha_k\rangle = \frac{A\,\mathrm{e}^{9K'}+9\,\mathrm{e}^{5K'}+27\,\mathrm{e}^{K'}+27\,\mathrm{e}^{-K'}}{8\,\mathrm{e}^{9K'}+72\,\mathrm{e}^{5K'}+216\,\mathrm{e}^{K'}+216\,\mathrm{e}^{-K'}} \tag{28}$$

where

$$A = \frac{2+6\cosh^2(4K)}{[1+\cosh(4K)]^3} \tag{29}$$

which approaches 1 and then $\langle \alpha_i\alpha_j\alpha_k\rangle \to \frac{1}{8}$ as $K\to 0$. For higher correlation functions, such as $\langle \alpha_i\alpha_j\alpha_k\alpha_l\rangle$, $\langle \alpha_i\alpha_j\alpha_k\alpha_l\alpha_m\rangle,\ldots$ and $\langle \alpha_1\alpha_2\ldots\alpha_n\rangle$, for the similar calculation they give a contribution of $(\frac{1}{2})^4$, $(\frac{1}{2})^5,\ldots$ and $(\frac{1}{2})^n$ respectively and a K-dependent contribution. In the limit of $K\to 0$, the latter can be neglected.

Finally, we reach the following result which is a high-temperature approximation,

$$\frac{Z_{\mathrm{SG}}}{Z_{\mathrm{DSG}}} = \left\langle \frac{A_1}{B_1}\frac{A_2}{B_2}\cdots\frac{A_n}{B_n}\right\rangle_{\mathrm{DSG}}$$
$$= (\tfrac{1}{2})^n + 3^{N-2}(\tfrac{1}{2}K^2+2K^4)^3(1+9c(K')+27c^2(K')+27c^3(K'))+\cdots. \tag{30}$$

We now prove that the expression (30) approaches to $(\frac{1}{2})^n$ when $N\to\infty$ and $K\to 0$. In fact, as $K\to 0$, for example,

$$\frac{A_1}{B_1} = \frac{\cosh K(s_2+s_4-s_7-s_{11})}{\cosh K(s_2+s_4+s_7+s_{11})+\cosh K(s_2+s_4-s_7-s_{11})} \to \frac{1}{2} \tag{31}$$

then

$$Z_{\mathrm{SG}}/Z_{\mathrm{DSG}} = (\tfrac{1}{2})^n. \tag{32}$$

6. Conclusion and discussion

We have proposed a method which can systematically calculate the partition function to any order of K. In the high-temperature approach the result shown in (30) is extremely precise.

In the appendix we list the expressions of A_i and B_i in 2^4 configurations of s_2, s_4, s_7 and s_{11}. It is found that in all configurations except $(s_2, s_4, s_7, s_{11}) = (1, 1, 1, 1)$, $(1, 1, -1, -1)$, $(-1, -1, 1, 1)$ and $(-1, -1, -1, -1)$ $A_i = \frac{1}{2}B_i$. Therefore we argue that $Z_{\mathrm{SG}} = (\frac{1}{2})^n Z_{\mathrm{DSG}}$ may be a very good approximation even if K is not too small. We expect that it will be confirmed further.

Our method can be easily generalised to a Sierpinski gasket embedded in a high-dimensional Euclidean space. In fact, for a fractal with a finite order of ramification we may calculate the partition function by introducing a corresponding decoupled structure.

Acknowledgment

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Appendix

In this appendix we list the expressions of A_i and B_i in 2^4 configurations of s_2, s_4, s_7 and s_{11}. It shows us that $A_i = \frac{1}{2}B_i$ in all configurations except $(s_2, s_4, s_7, s_{11}) = (1, 1, 1, 1)$, $(1, 1, -1, -1)$, $(-1, -1, 1, 1)$ and $(-1, -1, -1, -1)$, where

$$A_i = 2\cosh K(s_2 + s_4 - s_7 - s_{11})$$

and

$$B_i = 2\cosh K(s_2 + s_4 - s_7 - s_{11}) + 2\cosh K(s_2 + s_4 + s_7 + s_{11}).$$

Table 1 summarises these results.

Table 1.

Configurations					
s_2	s_4	s_7	s_{11}	$A_i/2$	$B_i/2$
* 1	1	1	1	1	$1+\cosh 4K$
1	1	1	−1	$\cosh 2K$	$2\cosh 2K$
1	1	−1	1	$\cosh 2K$	$2\cosh 2K$
* 1	1	−1	−1	$\cosh 4K$	$1+\cosh 4K$
1	−1	1	1	$\cosh 2K$	$2\cosh 2K$
1	−1	1	−1	1	2
1	−1	−1	1	1	2
1	−1	−1	−1	$\cosh 2K$	$2\cosh 2K$
−1	1	1	1	$\cosh 2K$	$2\cosh 2K$
−1	1	1	−1	1	2
−1	1	−1	1	1	2
−1	1	−1	−1	$\cosh 2K$	$2\cosh 2K$
*−1	−1	1	1	$\cosh 4K$	$1+\cosh 4K$
−1	−1	1	−1	$\cosh 2K$	$2\cosh 2K$
−1	−1	−1	1	$\cosh 2K$	$2\cosh 2K$
*−1	−1	−1	−1	1	$1+\cosh 4K$

* shows that $A_i \neq \frac{1}{2}B_i$

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